Orbit Stabilizer Theorem -

Let the group G act on the set X. Let $x \in X$
Then the orbit Gx is in 1-1 correspondence
with G/G_x
set of cosets

in particular, if G is finite,

$$\frac{|G|}{|G_x|} = |Gx|$$

For $x \in X$
$G_x = \{g \in G : gx = x\} < G$
$Gx = \{gx : g \in G\} \subseteq X$

Proof: with notation as above, we define

$$F: G/G_x \to Gx \text{ by}$$

$$F(aG_x) = ax.$$

First, we show this function is well-defined -

$F: P \to Q$ is well-defined if $p_1 = p_2$, then $F(p_1) = F(p_2)$.

Suppose $aG_x = bG_x \Leftrightarrow b^{-1}aG_x = G_x$
$\Leftrightarrow b^{-1}a \in G_x$. $\Leftrightarrow b^{-1}a = g$ for some $g \in G_x$.

then $bx = bgx = bb^{-1}ax = ax$. ✓

Next, observe that F is onto, because, $\forall hx \in Gx$,

$$hx = F(hG_x).$$

Also, $\forall\, aG_x, bG_x \in G/G_x$, if
$F(aG_x) = F(bG_x)$, then $ax = bx \Rightarrow b^{-1}ax = x$
$\Rightarrow b^{-1}a \in G_x \underset{\text{above}}{\Leftrightarrow} aG_x = bG_x$. $\therefore$ F is 1-1.

$\therefore$ F is a 1-1 correspondence.

Another important fact: Cayley's Thm.

If G is a finite group with n elements, then G is isomorphic to a subgroup of S_n.

Idea of proof.

Let $G = \{g_1, g_2, \dots, g_n\}$

$$\begin{array}{cccc} g_1 & g_2 & & g_n \\ \updownarrow & \updownarrow & & \updownarrow \\ 1 & 2 & & n \end{array}$$

Now we act on G by left multiplication

$\forall h \in G$, consider $\underbrace{g_j \longmapsto h g_j}_{\forall j} = g_{k_j}$ for some k_j

The isomorphism is

$$F(h) = \sigma \in S_n, \text{ where}$$
$$\sigma(j) = k_j \;\; \forall j.$$

(Note with this action $G g_j = G$

$$\Rightarrow G_{g_j} = \{e\})$$

This function $F : G \to S_n$ is a homomorphism (we can check that), and it is 1-1, so it is an isomorphism onto $F(G) \leq S_n$.

Example: Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\overset{g_1}{(0,0)}, \overset{g_2}{(1,0)}, \overset{g_3}{(0,1)}, \overset{g_4}{(1,1)}\}$.

$\qquad\qquad 1 \qquad 2 \qquad 3 \qquad 4$

$F((0,0)) = e \in S_4$

$F((1,0)) = (1,2)(3,4)$

$(1,0) + (0,0)^1 = (1,0)^2$

$(1,0) + (1,0)^2 = (0,0)^1$

$(1,0) + (0,1)^3 = (1,1)^4$

$(1,0) + (1,1)^4 = (0,1)^3$

$F((0,1)) = (1,3)(2,4)$

$(0,1) + (0,0)^1 = (0,1)^3$
$(0,1) + (1,0)^2 = (1,1)^4$
$(0,1) + (0,1)^3 = (0,0)^1$
$(0,1) + (1,1)^4 = (1,0)^2$

$F((1,1)) = (1,4)(2,3)$

$(1,1) + (0,0)^1 = (1,1)^4$
$(1,1) + (1,0)^2 = (0,1)^3$
$(1,1) + (0,1)^3 = (1,0)^2$
$(1,1) + (1,1)^4 = (0,0)^1$

$\{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$

Definitions: Let G be a group, and let $g \in G$, let $S \subseteq G$.
S: Not necessarily a subgroup.

1. $C_G(g) =$ Centralizer of g in G
$= \{h \in G : gh = hg\}$ $\quad h^{-1}gh = g$
$=$ Normalizer of $\{g\} = N_G(\{g\})$.

2. $C_G(S) =$ Centralizer of S in G
$= \{h \in G : gh = hg \text{ for all } g \in S\}$. $\quad h^{-1}gh = g$.

3. $N_G(S) = \{h \in G : hSh^{-1} = S\}$
S: usually a subgroup
$=$ Normalizer of S in G.

4. $Z(G) = \{h \in G : hx = xh \ \forall x \in G\}$
$=$ Center of G.

Notice for any subset $S \subseteq G$,

$$Z(G) \subseteq C_G(S) \subseteq N_G(S) \subseteq G$$

Important fact: All of these are subgroups

Example proof: Prove that if $S \subseteq G$, then $N_G(S)$ is a subgroup of G.

maybe we need S to be a subgroup!

Pf: Let $e \in G$ be the identity.

then $eSe^{-1} = eSe = S$

$\Rightarrow e \in N_G(S)$.

Suppose $g_1, g_2 \in N_G(S)$, $h \in S$

Then $(g_1 g_2) S (g_1 g_2)^{-1}$

$$= g_1 g_2 S g_2^{-1} g_1^{-1} = g_1 \underbrace{\left(g_2 S g_2^{-1}\right)}_{S \text{ since } g_2 \in N_G(S)} g_1^{-1}$$

$$= S$$

S since $g_1 \in N_G(S)$

$\therefore g_1 g_2 \in N_G(S)$.

Suppose $g \in N_G(S)$, then $gSg^{-1} = S$

$\Rightarrow gS = Sg \Rightarrow S = g^{-1}Sg \Rightarrow g^{-1} \in N_G(S)$